

BLOW UP PROPERTY FOR VISCOELASTIC EVOLUTION EQUATIONS ON MANIFOLDS WITH CONICAL DEGENERATION

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ABSTRACT. This paper is concerned with the study of the nonlinear viscoelastic evolution equation with strong damping and source terms, described by

$$u_{tt} - \Delta_{\mathbb{B}} u + \int_0^t g(t - \tau) \Delta_{\mathbb{B}} u(\tau) d\tau + f(x) u_t |u_t|^{m-2} = h(x) |u|^{p-2} u, \quad x \in \text{int} \mathbb{B}, t > 0,$$

where \mathbb{B} is a stretched manifold. First, we prove the solutions of problem 1.1 in cone Sobolev space $\mathcal{H}_{2,0}^{1,\frac{p}{2}}(\mathbb{B})$, admit a blow up in finite time for $p > m$ and positive initial energy. Then, we construct a lower bound for obtained blow up time under appropriate assumptions on data.

1. INTRODUCTION

Nonlinear differential equations and their solutions play an important role at description of physical and other processes. For instance, a purely elastic material has a capacity to store mechanical energy with no dissipation of the energy. A complete opposite to an elastic material is a purely viscous material. The important point about viscous materials is that when the force is removed it does not return to its original shape. Materials which are outside the scope of these two models will be those for which some of the work done to deform them can be recovered. Such materials possess a capacity of storage and dissipation of mechanical energy. This is the case for viscoelastic material. It is well known that viscoelastic materials exhibit natural damping, which is due to the special property of these materials to retain a memory of their past history [1, 2]. It is worth mentioning the works in connection with viscoelastic effects on a domain Ω of \mathbb{R}^n . Cavalcanti et al. [3], firstly investigated

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t - \tau) \Delta u(\tau) d\tau + a(x) u_t + |u|^\gamma u = 0, & x \in \Omega, t > 0, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in \Omega \\ u(t, x) = 0, & x \in \partial\Omega, t \geq 0, \end{cases} \quad (1.1)$$

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and obtained an exponential decay rate of the solution under some assumptions on $g(s)$ and $a(x)$. It is important to mention some papers in connection with viscoelastic effects, among them, Alves and Cavalcanti [4], Aassila et.al [5] and references therein. Cavalcanti and Oquendo [6] studied

$$u_{tt} - k_0 \Delta u + \int_0^t \operatorname{div}[a(x)g(t-\tau)\nabla u]d\tau + b(x)h(u_t) + f(u) = 0,$$

under the restrictive assumptions on both the damping function h and the kernel g . Moreover, Messaoudi [7] studied the global existence of solutions for the viscoelastic equation, at the same time he also obtained a blow-up result with negative energy. Then, he improved his blow-up result in [8]. Recently, Song and Xue [9], considered the nonlinear viscoelastic equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau - \Delta u_t = |u|^{p-2}u \quad \text{on } \Omega \times [0, T],$$

where Ω is a bounded domain of \mathbb{R}^n with a smooth boundary $\partial\Omega$. They showed under suitable conditions on g , that there are solutions to their problem with arbitrarily high initial energy which blow up in finite time. Cavalcanti et al. [10] considered a nonlinear viscoelastic evolution equation as

$$u_{tt} + Au + F(x, t, u, u_t) - \int_0^t g(t-\tau)Au(\tau)d\tau = 0 \quad \text{on } \Gamma \times (0, \infty)$$

where Γ is a compact manifold. When $F \neq 0$ and $g \neq 0$ they proved the existence of global solutions as well as uniform decay rates. Furthermore, Cavalcanti et al. [11], studied the wave equation on compact surfaces and locally distributed damping, described by

$$u_{tt} - \Delta_{\mathcal{M}}u + a(x)g(u_t) = 0 \quad \text{on } \mathcal{M} \times (0, \infty),$$

where $\mathcal{M} \subset \mathbb{R}^3$ is a smooth oriented embedding compact surface without boundary and $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator on \mathcal{M} . Recently, the authors in [12], discussed the asymptotic stability of the wave equation on a compact Riemannian manifold subject to locally distributed viscoelastic effects. Also, they proved that the solutions of the corresponding partial viscoelastic model decay exponentially to zero. This paper motivated by two works. The first, analog to [11], we consider Fuchsian-Laplace operator $\Delta_{\mathbb{B}}$ as a special case of the typical differential operators on a manifold with conical singularity so-called Fuchs type operator. The second, using of the nonlinear viscoelastic evolution equation in [10], we will investigate a nonlinear viscoelastic evolution problem includes Fuchsian-Laplace operator, strong damping and source terms. More precisely, we want to study blow up of solutions a nonlinear viscoelastic problem contains Fuchsian-Laplace operator and nonlinear damping and source terms on a manifold with conical singularity points. Moreover, when blow up occurs, the blow up time T cannot

usually be computed exactly. We note that, in general, this is very difficult to obtain a lower bound for viscoelastic wave equation. Hence, it is very attractive and interesting subject of inquiry to specify lower and upper bounds for blow up time T . From the mathematical point of view, the damping effects are modeled by integro-differential operators. Therefore, the dynamics of viscoelastic materials are of great importance and interest as they have wide applications in natural sciences. Thereupon, we consider an initial-boundary value problem for a nonlinear viscoelastic wave equation with strong damping, nonlinear damping and source terms on a manifold with conical degenerations as follows:

$$\begin{cases} u_{tt} - \Delta_{\mathbb{B}} u + \int_0^t g(t-\tau) \Delta_{\mathbb{B}} u(\tau) d\tau + f(x) u_t |u_t|^{m-2} = h(x) |u|^{p-2} u, & x \in \text{int}\mathbb{B}, t > 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \text{int}\mathbb{B} \\ u(t, x) = 0, & x \in \partial\mathbb{B}, t \geq 0, \end{cases} \quad (1.2)$$

where $p > 2$, $m \geq 2$, and g is a C^1 -function and we assume that the function g satisfies the following conditions:

$A_1)$ $1 - \int_0^\infty g(s) ds = l > 0$. This assumption guarantee the hyperbolicity and well-posedness of the system 1.2.

$A_2)$ $g(s) \geq 0$ and $g'(s) \leq 0$;

$A_3)$ $\int_0^\infty g(s) ds < 1 - \frac{1}{(p-1)^2}$;

$A_4)$ $f, h \in L^\infty(\text{int}\mathbb{B}) \cap C(\text{int}\mathbb{B})$ are positive functions.

The domain \mathbb{B} equals $[0, 1) \times X$, X is an $(n-1)$ -dimensional closed compact manifold ($n \geq 3$), which is regarded as the local model near the conical points on manifolds with conical singularities, and $\partial\mathbb{B} = \{0\} \times X$. Also, we assume that the volume of \mathbb{B} is finite, i.e. $|\mathbb{B}| = \int_{\mathbb{B}} \frac{dx_1}{x_1} dx' < +\infty$. Moreover, the operator $\Delta_{\mathbb{B}}$ in 1.2 is defined by $(x_1 \partial_{x_1})^2 + \partial_{x_2}^2 + \dots + \partial_{x_n}^2$, which is an elliptic operator with totally characteristic degeneracy on the boundary $x_1 = 0$, we also call it Fuchsian type Laplace operator, and the corresponding gradient operator by $\nabla_{\mathbb{B}} := (x_1 \partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n})$. Near $\partial\mathbb{B}$ we will often use coordinates $(x_1, x') = (x_1, x_2, \dots, x_n)$ for $x_1 \in [0, 1)$ and $x' \in X$. To more details about the manifold with singularities and elliptic differential operators with totally characteristic degeneration see Section 2 and the references [17, 18].

In the absence of the viscoelastic term ($g = 0$), the problem 1.2 has been extensively studied and results concerning existence and nonexistence have been established in Euclidean domains.

In bounded domains, for the equation

$$u_{tt} - \Delta u + a|u_t|^{m-2}u_t = b|u|^{p-2}u \quad \text{on } \Omega \times (0, \infty),$$

where $m \geq 2$, $p > 2$ and a, b are positive constants, the interaction between the damping and source terms was first considered by Levine [13, 14] for the linear damping case $m = 2$ and also the author proved that solutions with negative initial energy blow up in finite time. Georgiev and Todorova [15] investigated Levine's results for the nonlinear damping case $m > 2$ and they proved that solutions with any initial data continue to exist globally if $m \geq p$ and blow up in finite time if $p > m$ and the initial energy is sufficiently negative. For the problem 1.2 without viscoelastic and damping terms on a manifold with conical singularity, the authors [16] used the cone Sobolev and Poincaré inequalities to prove the existence results of global solutions with exponential decay and showed the blow up in finite time of solutions in cone Sobolev space $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$.

Suppose that C_{emb} is the best constant of the Sobolev embedding $\mathcal{H}_{2,0}^{1,\frac{n}{2}} \hookrightarrow L^{\frac{n}{p}}$ for $p \in [2, 2^*]$ and

$$C_* := \frac{C_{emb}C_{Poin}}{l^{\frac{1}{2}}}, \quad (1.3)$$

where C_{Poin} is the Poincaré constant. Furthermore, we assume that

$$C^* = \inf \left\{ \frac{\int_{\mathbb{B}} f(x) |u_t|^m \frac{dx_1}{x_1} dx'}{\|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{m}}(\mathbb{B})}^m} : u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \right\}, \quad (1.4)$$

and

$$\max \{ m, p \} \leq \frac{2(n-1)}{n-2} \quad \forall n \geq 3. \quad (1.5)$$

Now, we state our main results as follows. In the first result, we conclude blow up for solutions of problem 1.2 and in the second result, we present a lower bound for provided blow up time T under suitable conditions on components of our problem.

Theorem 1.1. *Assume that the conditions $A_1 - A_4$ hold. Let p and m be such that $m \geq 2$ and $p > \max\{2, m\}$. Then, any weak solution of 1.2 with initial data satisfying the following conditions blows up in finite time.*

$$I(0) < I_1, \quad \|\nabla_{\mathbb{B}} u_0\|_{L^{\frac{n}{2}}(\mathbb{B})} > (C_* \sqrt[p]{C_h})^{-\frac{p}{p-2}}.$$

Theorem 1.2. *Suppose that the assumptions in Theorem 1.1 hold. Further, we deem that $u(x, t)$ is a weak solution of 1.2, which blows up at a finite time T . Then, T admits a lower bound as follows*

$$\int_{\mathcal{H}(0)}^{\infty} \frac{1}{\mathcal{C}_3 S^k + S + \mathcal{C}_4} ds \leq T.$$

Our paper organized as follows. Section 2 is devoted to introduce and construct stretched manifold \mathbb{B} associated to a given manifold with conical singularities. Section 3 is concerned with some subsidiary results about the energy functional corresponding to the problem 1.2. Finally, section 4 is dedicated to proof of our main results.

2. CONE SOBOLEV SPACES

In this section, we recall some definitions and notations from Sobolev spaces on manifolds with conical singularities. We refer enthusiastic readers to [17, 18] and the references therein.

Let B be a manifold with conical singularities x_1, \dots, x_N . First, for simplicity let us consider the case $N = 1$ and set $x = x_1$. If X is C^∞ closed compact manifold, the cone $X^\Delta := \frac{\mathbb{R}_+ \times X}{\{0\} \times X}$ is an example of such an B . In this case the conical singularity is represented by $\{0\} \times X$ in the quotient space. In general, B is locally near x modelled on such a cone. More precisely, $B - \{x\}$ is smooth, and we have a singular chart

$$\chi : G \rightarrow X^\Delta$$

for some neighborhood G of x in M and a smooth manifold $X = X(x)$ where $\chi(x)$ is equal to the vertex of X^Δ , $\varphi = \chi|_{G - \{x\}} : G - \{x\} \rightarrow X^\Delta := \mathbb{R}_+ \times X$ is a diffeomorphism [17]. More precisely, a finite dimensional manifold B with conical singularities, is a topological space with a finite subset $B_0 = \{x_1, \dots, x_N\} \subset B$ of conical singularities, which has the following two properties:

- a) $B - B_0$ is a C^∞ manifold.
- b) any $x \in B_0$ has an open neighborhood G in B , such that there is a homeomorphism $\chi : G \rightarrow X^\Delta$ for some closed compact C^∞ manifold $X = X(x)$ and φ restricts a diffeomorphism $\varphi' : G - \{x\} \rightarrow X^\Delta$.

For such a manifold, let $n \geq 2$ and $X \subset S^{n-1}$ be a bounded open set in the unit sphere of \mathbb{R}^n_x . The set $B := \left\{ x \in \mathbb{R}^n - \{0\} ; \quad \frac{x}{|x|} \in X \right\} \cup \{0\}$ is an infinite cone with the base X and the critical point $\{0\}$. Using the polar coordinates, one can get a description of $B - \{0\}$ in the

form $X^\wedge = \mathbb{R}_+ \times X$, which is called the open stretched cone with the base X , and $\{0\} \times X$ is the boundary of X^\wedge .

Now, we assume that the manifold B is paracompact and of dimension n . By this assumption we can define the stretched manifold associated with B . Let \mathbb{B} be a C^∞ manifold with compact C^∞ boundary $\partial\mathbb{B} \cong \bigcup_{x \in B_0} X(x)$ for which there exists a diffeomorphism $B - B_0 \cong \mathbb{B} - \partial\mathbb{B} := \text{int}\mathbb{B}$, the restriction of which to $G_1 - B_0 \cong U_1 - \partial\mathbb{B}$ for an open neighborhood $G_1 \subset B$ near the points of B_0 and a collar neighborhood $U_1 \subset \mathbb{B}$ with $U_1 \cong \bigcup_{x \in B_0} \{[0, 1) \times X(x)\}$. The typical differential operators on a manifold with conical singularities, called Fuchs type, are operators that are in a neighborhood of $x_1 = 0$ of the following form

$$A = x_1^{-m} \sum_{k=0}^m a_k(x_1) (-x_1 \partial_{x_1})^k$$

with $(x_1, x) \in X^\wedge$ and $a_k(x_1) \in C^\infty(\bar{\mathbb{R}}_+, \text{Diff}^{m-k}(X))$ [18]. The differential $x_1 \partial_{x_1}$ in Fuchs type operators provokes us to apply the Mellin transform $M : C_0^\infty(\mathbb{R}_+) \rightarrow \mathcal{A}(\mathbb{C})$, for $u(x_1) \in C_0^\infty(\mathbb{R}_+)$, $z \in \mathbb{C}$, defined as

$$Mu(z) := \int_0^{+\infty} x_1^z u(x_1) \frac{dx_1}{x_1}, \quad (2.1)$$

where $\mathcal{A}(\mathbb{C})$ denotes the space of entire functions.

One can find further details on Fuchs type operators and all implications and definitions of the cone Sobolev spaces in [17, 18]. We use the so-called weighted Melline transform

$$M_\gamma u := Mu|_{\Gamma_{\frac{1}{2}-\gamma}} = \int_0^{+\infty} x_1^{\frac{1}{2}-\gamma+i\tau} u(x_1) \frac{dx_1}{x_1},$$

where $\Gamma_\beta := \{z \in \mathbb{C} ; \text{Re} z = \beta\}$. The inverse weighted Melline transform is defined as

$$(M_\gamma^{-1}g)(x_1) = \frac{1}{2i\pi} \int_{\Gamma_{\frac{1}{2}}} x_1^{-z} g(z) dz.$$

In order to define cone Sobolev spaces on the stretched manifolds, at the first we introduce the weighted Sobolev spaces on \mathbb{R}^n and then by using of partition unity, we introduce suitable weighted cone Sobolev space on the stretched manifold \mathbb{B} .

Definition 2.1. For $(x_1, x') \in \mathbb{R}_+ \times \mathbb{R}^{n-1} = \mathbb{R}_+^n$ we say that $u(x_1, x') \in L_p(\mathbb{R}_+^n, \frac{dx_1}{x_1} dx')$ if

$$\|u\|_{L_p} = \left(\int_{\mathbb{R}_+} \int_{\mathbb{R}^{n-1}} x_1^n |u(x_1, x')|^p \frac{dx_1}{x_1} dx' \right)^{\frac{1}{p}} < \infty.$$

The weighted L_p -spaces with weight data $\gamma \in \mathbb{R}$ is denoted by $L_p^\gamma(\mathbb{R}_+^n, \frac{dx_1}{x_1} dx')$. In fact, if $u(x_1, x') \in L_p^\gamma(\mathbb{R}_+^n, \frac{dx_1}{x_1} dx')$, then $x_1^{-\gamma} u(x_1, x') \in L_p(\mathbb{R}_+^n, \frac{dx_1}{x_1} dx')$, and

$$\|u\|_{L_p^\gamma}^\gamma = \left(\int_{\mathbb{R}_+} \int_{\mathbb{R}^{n-1}} x_1^n |x_1^{-\gamma} u(x_1, x')|^p \frac{dx_1}{x_1} dx' \right)^{\frac{1}{p}} < \infty.$$

Now, we can define the weighted p -Sobolev spaces for $1 \leq p < \infty$.

Definition 2.2. For $m \in \mathbb{N}$, $\gamma \in \mathbb{R}$ and $1 \leq p < \infty$, the spaces

$$\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^n) := \left\{ u \in \mathcal{D}'(\mathbb{R}_+^n) ; x_1^{\frac{n}{p}-\gamma} (x_1 \partial_{x_1})^\alpha \partial_{x'}^\beta u \in L_p(\mathbb{R}_+^n, \frac{dx_1}{x_1} dx') \right\}, \quad (2.2)$$

for any $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}^{n-1}$ and $|\alpha| + |\beta| \leq m$. In other words, if $u(x_1, x) \in \mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^n)$, then $(x_1 \partial_{x_1})^\alpha \partial_{x'}^\beta u \in L_p^\gamma(\mathbb{R}_+^n, \frac{dx_1}{x_1} dx')$.

Hence, $\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^n)$ is a Banach space with norm

$$\|u\|_{\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^n)} = \sum_{|\alpha|+|\beta| \leq m} \left(\int_{\mathbb{R}_+^n} x_1^n |x_1^{-\gamma} (x_1 \partial_{x_1})^\alpha \partial_{x'}^\beta u(x_1, x')|^p \frac{dx_1}{x_1} dx' \right)^{\frac{1}{p}}.$$

Let X be a closed compact C^∞ manifold, and $\mathcal{U} = \{U_1, \dots, U_N\}$ an open covering of X by coordinate neighborhoods. If we fix a subordinate partition of unity $\{\varphi_1, \dots, \varphi_N\}$ and charts $\chi_j : U_j \rightarrow \mathbb{R}^{n-1}$, $j = 1, \dots, N$. Then we say that $u \in \mathcal{H}_p^{m,\gamma}(X^\wedge)$ if and only if $u \in \mathcal{D}'(X^\wedge)$ with the norm

$$\|u\|_{\mathcal{H}_p^{m,\gamma}(X^\wedge)} = \left\{ \sum_{j=1}^N \|(1 \times \chi_j^*)^{-1} \varphi_j u\|_{\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^n)}^p \right\}^{\frac{1}{p}} < \infty,$$

where $1 \times \chi_j^* : C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^{n-1}) \rightarrow C_0^\infty(\mathbb{R}_+ \times U_j)$ is the pull-back function with respect to $1 \times \chi_j : \mathbb{R}_+ \times U_j \rightarrow \mathbb{R}_+ \times \mathbb{R}^{n-1}$. Denote the $\mathcal{H}_{p,0}^{m,\gamma}(X^\wedge)$ as subspace of $\mathcal{H}_p^{m,\gamma}(X^\wedge)$ which is defined as the closure of $C_0^\infty(X^\wedge)$ with respect to the norm $\|\cdot\|_{\mathcal{H}_p^{m,\gamma}(X^\wedge)}$. Now, we have the following definition

Definition 2.3. Let $\mathbb{B} = [0, 1) \times X$ be the stretched manifold of the manifold B with conical singularity. Then the cone Sobolev space $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$, for $m \in \mathbb{N}$, $\gamma \in \mathbb{R}$ and $p \in (1, \infty)$, is defined by

$$\mathcal{H}_p^{m,\gamma}(\mathbb{B}) = \{u \in W_{loc}^{m,p}(\text{int}\mathbb{B}) ; \omega u \in \mathcal{H}_p^{m,\gamma}(X^\wedge)\},$$

for any cut-off function ω supported by a collar neighborhood of $[0, 1) \times \partial\mathbb{B}$. Moreover, the subspace $\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B})$ of $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$ is defined by

$$\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B}) := (\omega) \mathcal{H}_{p,0}^{m,\gamma}(X^\wedge) + (1 - \omega) W_0^{m,p}(\text{int}\mathbb{B}),$$

where $W_0^{m,p}(int\mathbb{B})$ denotes the closure of $C_0^\infty(int\mathbb{B})$ in Sobolev space $W^{m,p}(\tilde{X})$ when \tilde{X} is closed compact C^∞ manifold of dimension n containing \mathbb{B} as a submanifold with boundary.

Definition 2.4. Let $\mathbb{B} = [0, 1) \times X$. We say that $u(x) \in L_p^\gamma(\mathbb{B})$ with $1 < p < \infty, \gamma \in \mathbb{R}$, if

$$\|u\|_{L_p^\gamma(\mathbb{B})} = \int_{\mathbb{B}} x_1^n |x_1^{-\gamma} u(x)|^p \left(\frac{dx_1}{x_1}\right) dx' < \infty.$$

For $\gamma = \frac{n}{p}$ and $\gamma = \frac{n}{q}$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have the following Hölders inequality

$$\int_{\mathbb{B}} |u(x)v(x)| \frac{dx_1}{x_1} dx' \leq \left(\int_{\mathbb{B}} |u(x)|^p \frac{dx_1}{x_1} dx' \right)^{\frac{1}{p}} \left(\int_{\mathbb{B}} |v(x)|^q \frac{dx_1}{x_1} dx' \right)^{\frac{1}{q}}. \quad (2.3)$$

In the sequel, for convenience we denote

$$(u, v)_2 = \int_{\mathbb{B}} u(x)v(x) \frac{dx_1}{x_1} dx' \text{ and } \|u\|_{L_p^{\frac{n}{p}}(\mathbb{B})} = \int_{\mathbb{B}} |u(x)|^p \frac{dx_1}{x_1} dx'.$$

3. SOME AUXILIARY RESULTS

In this section, we give some auxiliary results about the problem 1.2 and we get some properties of energy functional that we will use to prove our main results. therefore, similar to the classical case, we introduced functional $I(t) = I(u(t), u_t(t))$, on cone Sobolev space $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ as follows:

$$\begin{aligned} I(t) &= \frac{1}{2} \|u_t(t)\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + \frac{1}{2} \left(1 - \int_0^t g(s) ds \right) \|\nabla_{\mathbb{B}} u(t)\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + \frac{1}{2} (g \circ \nabla_{\mathbb{B}} u)(t) \\ &\quad - \frac{1}{p} \int_{\mathbb{B}} h(x) |u|^p \frac{dx_1}{x_1} dx', \end{aligned} \quad (3.1)$$

where

$$(g \circ v)(t) = \int_0^t g(t - \tau) \|v(t) - v(\tau)\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 d\tau.$$

Lemma 3.1. Suppose that conditions $A_1 - A_4$ and 1.5 hold. Let u be a solution of the problem 1.2. Then the energy functional $I(t)$ is non-increasing.

Proof. Let us multiply Eq. (1.1) by u_t and integrating over \mathbb{B} , then e obtain

$$\begin{aligned} \langle u_{tt}, u_t \rangle &+ \langle -\Delta_{\mathbb{B}} u, u_t \rangle + \left\langle \int_0^t g(t - \tau) \Delta_{\mathbb{B}} u(\tau) d\tau, u_t \right\rangle + \langle f(x) u_t |u_t|^{m-2}, u_t \rangle \\ &= \langle h(x) u |u|^{p-2}, u_t \rangle \Rightarrow \end{aligned} \quad (3.2)$$

$$\begin{aligned} \frac{d}{dt} \left[\frac{1}{2} \int_{\mathbb{B}} |u_t|^2 \frac{dx_1}{x_1} dx' + \frac{1}{2} \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1} dx' - \frac{1}{p} \int_{\mathbb{B}} h(x) |u|^p \frac{dx_1}{x_1} dx' \right] \\ - \int_0^t g(t - \tau) \int_{\mathbb{B}} \nabla_{\mathbb{B}} u_t(t) \cdot \nabla_{\mathbb{B}} u(\tau) \frac{dx_1}{x_1} dx' d\tau = - \int_{\mathbb{B}} f(x) |u_t|^m \frac{dx_1}{x_1} dx'. \end{aligned} \quad (3.3)$$

On the other hand,

$$\begin{aligned}
\int_0^t g(t-\tau) \int_{\mathbb{B}} \nabla_{\mathbb{B}} u_t(t) \cdot \nabla_{\mathbb{B}} u(\tau) \frac{dx_1}{x_1} dx' d\tau &= \int_0^t g(t-\tau) \int_{\mathbb{B}} \nabla_{\mathbb{B}} u_t(t) \cdot (\nabla_{\mathbb{B}} u(\tau) - \nabla_{\mathbb{B}} u(t)) \frac{dx_1}{x_1} dx' d\tau \\
&+ \int_0^t g(t-\tau) \int_{\mathbb{B}} \nabla_{\mathbb{B}} u_t(t) \cdot \nabla_{\mathbb{B}} u(t) \frac{dx_1}{x_1} dx' d\tau \\
&= -\frac{1}{2} \int_0^t g(t-\tau) \frac{d}{dt} \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u(\tau) - \nabla_{\mathbb{B}} u(t)|^2 \frac{dx_1}{x_1} dx' d\tau \\
&+ \int_0^t g(\tau) \left(\frac{d}{dt} \frac{1}{2} \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u(t)|^2 \frac{dx_1}{x_1} dx' \right) d\tau \\
&= -\frac{1}{2} \frac{d}{dt} \left[\int_0^t g(t-\tau) \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u(\tau) - \nabla_{\mathbb{B}} u(t)|^2 \frac{dx_1}{x_1} dx' d\tau \right] \\
&+ \frac{1}{2} \frac{d}{dt} \left[\int_0^t g(\tau) \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u(t)|^2 \frac{dx_1}{x_1} dx' d\tau \right] \\
&+ \frac{1}{2} \int_0^t g'(t-\tau) \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u(\tau) - \nabla_{\mathbb{B}} u(t)|^2 \frac{dx_1}{x_1} dx' d\tau \\
&- \frac{1}{2} g(t) \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u(t)|^2 \frac{dx_1}{x_1} dx'.
\end{aligned} \tag{3.4}$$

Now, we situate estimations 3.4 in the relation 3.3, we get

$$\begin{aligned}
\frac{d}{dt} \left[\frac{1}{2} \int_{\mathbb{B}} |u_t|^2 \frac{dx_1}{x_1} dx' \right] &+ \frac{1}{2} \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1} dx' - \frac{1}{p} \int_{\mathbb{B}} h(x) |u|^p \frac{dx_1}{x_1} dx' \\
&+ \frac{1}{2} \frac{d}{dt} \left[\int_0^t g(t-\tau) \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u(\tau) - \nabla_{\mathbb{B}} u(t)|^2 \frac{dx_1}{x_1} dx' d\tau \right] \\
&- \frac{1}{2} \frac{d}{dt} \left[\int_0^t g(\tau) \|\nabla_{\mathbb{B}} u(t)\|^2 d\tau \right] \\
&= - \int_{\mathbb{B}} f(x) |u_t|^m \frac{dx_1}{x_1} dx' + \frac{1}{2} \int_0^t g'(t-\tau) \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u(\tau) - \nabla_{\mathbb{B}} u(t)|^2 \frac{dx_1}{x_1} dx' d\tau \\
&- \frac{1}{2} g(t) \|\nabla_{\mathbb{B}} u\|_{L^{\frac{p}{2}}(\mathbb{B})}^2 \leq 0.
\end{aligned} \tag{3.5}$$

Therefore, $I'(t) \leq 0$. It follows that $I(t)$ is non-increasing. \square

Lemma 3.2. *Let u be a weak solution of 1.2 such that conditions $A_1 - A_4$ and relation 1.5 hold. Moreover, assume that*

$$I(0) < I_1, \quad \|\nabla_{\mathbb{B}} u_0\|_{L^{\frac{p}{2}}(\mathbb{B})} > (C_* \sqrt[p]{C_h})^{-\frac{p}{p-2}}. \tag{3.6}$$

Then, there exists a constant $\beta > (C_ \sqrt[p]{C_h})^{-\frac{p}{p-2}}$ such that for any $t \in [0, T)$*

$$\left[\frac{1}{2} \left(1 - \int_0^t g(s) ds \right) \|\nabla_{\mathbb{B}} u(t)\|_{L^{\frac{p}{2}}(\mathbb{B})}^2 + \frac{1}{2} (g \circ \nabla_{\mathbb{B}} u)(t) \right]^{\frac{1}{2}} \geq \beta \tag{3.7}$$

and for every $t \in [0, T)$

$$\|u\|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} \geq \frac{\beta}{\sqrt{l}}. \quad (3.8)$$

Proof. According to definition of the functional I ,

$$\begin{aligned} I(t) &= \frac{1}{2} \|u_t(t)\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + \frac{1}{2} \left(1 - \int_0^t g(s) ds\right) \|\nabla_{\mathbb{B}} u(t)\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + \frac{1}{2} (g \circ \nabla_{\mathbb{B}} u)(t) \\ &- \frac{1}{p} \int_{\mathbb{B}} h(x) |u|^p \frac{dx_1}{x_1} dx' \geq \frac{1}{2} \left(1 - \int_0^t g(s) ds\right) \|\nabla_{\mathbb{B}} u(t)\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + \frac{1}{2} (g \circ \nabla_{\mathbb{B}} u)(t) \\ &- \frac{1}{p} \int_{\mathbb{B}} h(x) |u|^p \frac{dx_1}{x_1} dx'. \end{aligned} \quad (3.9)$$

On the other hand, by relation 1.3 and Hölder inequality we can obtain the following to estimations 3.9

$$\begin{aligned} &\frac{1}{2} \left(1 - \int_0^t g(s) ds\right) \|\nabla_{\mathbb{B}} u(t)\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + \frac{1}{2} (g \circ \nabla_{\mathbb{B}} u)(t) \\ &- \frac{C_*^p l^p C_h}{p} \|u\|_{\mathcal{H}_2^{1,\frac{n}{2}}(\mathbb{B})}^p \geq \frac{1}{2} \left(1 - \int_0^t g(s) ds\right) \|\nabla_{\mathbb{B}} u(t)\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + \frac{1}{2} (g \circ \nabla_{\mathbb{B}} u)(t) \\ &- \frac{C_*^p C_h}{p} \left[\left(1 - \int_0^t g(s) ds\right) \|\nabla_{\mathbb{B}} u(t)\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + \frac{1}{2} (g \circ \nabla_{\mathbb{B}} u)(t) \right]^{\frac{p}{2}}. \end{aligned} \quad (3.10)$$

If in relation 3.10, we take $\left[\frac{1}{2} \left(1 - \int_0^t g(s) ds\right) \|\nabla_{\mathbb{B}} u(t)\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + \frac{1}{2} (g \circ \nabla_{\mathbb{B}} u)(t) \right]^{\frac{1}{2}} = \theta$ then we have a function

$$k(\theta) = \frac{1}{2} \theta^2 - \frac{C_*^p C_h}{p} \theta^p. \quad (3.11)$$

By simple calculations, one can get

$$\max_{\theta} k(\theta) = k(\alpha),$$

where

$$\alpha := \left(\frac{1}{C_*^p C_h} \right)^{\frac{1}{p-2}}. \quad (3.12)$$

Then for any θ

$$k(\alpha) = \left(\frac{1}{C_*^p C_h} \right)^{\frac{2}{p-2}} \left[\frac{1}{2} - \frac{C_*^p C_h}{p} \right] \geq k(\theta). \quad (3.13)$$

Hence, for any $0 < \theta < \alpha$, $k(\theta)$ is increasing and for any $\theta > \alpha$, $k(\theta)$ is decreasing. It follows that as $\theta \rightarrow \infty$, $k(\theta) \rightarrow -\infty$ and

$$k(\alpha) = \left(\frac{1}{C_*^p C_h} \right)^{\frac{2}{p-2}} \left[\frac{1}{2} - \frac{C_*^p C_h}{p} \right] = I_1 \quad (3.14)$$

where α is given by 3.12. Using of the assumption $I(0) < I_1$ there exists a constant $\beta > \alpha$ for which $k(\beta) = I(0)$. Now, we set $\alpha_0 = \|\nabla_{\mathbb{B}} u_0\|_{L^{\frac{n}{2}}(\mathbb{B})}$, then by 3.11

$$\begin{aligned} k(\alpha_0) &= \frac{1}{2}\alpha_0^2 - \frac{C_*^p C_h}{p}\alpha_0^p = \frac{1}{2}\|\nabla_{\mathbb{B}} u_0\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 - \frac{C_*^p C_h}{p}\|\nabla_{\mathbb{B}} u_0\|_{L^{\frac{n}{2}}(\mathbb{B})}^p \\ &\leq I(0) = k(\beta). \end{aligned} \quad (3.15)$$

It follows that $\alpha_0 > \beta$.

Now, we use contradiction method to show that relation 3.7 is satisfied. Suppose that there exist some $t_0 > 0$ such that

$$\left[\left(1 - \int_0^{t_0} g(s)ds\right) \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + (g \circ \nabla_{\mathbb{B}} u)(t_0) \right]^{\frac{1}{2}} < \beta. \quad (3.16)$$

Since

$$\left(1 - \int_0^{t_0} g(s)ds\right) \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + (g \circ \nabla_{\mathbb{B}} u)(t_0)$$

is continuous, so one can choose t_0 for which

$$\left[\left(1 - \int_0^{t_0} g(s)ds\right) \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + (g \circ \nabla_{\mathbb{B}} u)(t_0) \right]^{\frac{1}{2}} > \alpha. \quad (3.17)$$

Using of relations 3.11 and 3.13, one can obtain

$$I(t_0) \geq k\left(\left[\left(1 - \int_0^{t_0} g(s)ds\right) \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + (g \circ \nabla_{\mathbb{B}} u)(t_0)\right]^{\frac{1}{2}}\right) > k(\beta) = I(0).$$

But, this is contradiction because $I(t) \leq I(0)$ for any $t \in [0, T)$. Therefore, the relation 3.7 is satisfied.

Now, we prove the relation 3.8. According to the definition of the energy functional $I(t)$ and Hölder inequality we have,

$$\begin{aligned} \frac{1}{2} \left[\left(1 - \int_0^{t_0} g(s)ds\right) \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + (g \circ \nabla_{\mathbb{B}} u)(t_0) \right]^{\frac{1}{2}} &\leq I(0) + \frac{1}{p} \int_{\mathbb{B}} h(x) |u|^p \frac{dx_1}{x_1} dx' \\ &\leq I(0) + \frac{C_h C_*^p l^{\frac{p}{2}}}{p} \|u\|_{\mathcal{H}_2^{1, \frac{n}{2}}(\mathbb{B})}^p. \end{aligned} \quad (3.18)$$

Therefore,

$$\begin{aligned} \frac{C_h C_*^p l^{\frac{p}{2}}}{p} \|u\|_{\mathcal{H}_{2,0}^{1, \frac{n}{2}}(\mathbb{B})}^p &\geq \frac{1}{2} \left[\left(1 - \int_0^{t_0} g(s)ds\right) \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + (g \circ \nabla_{\mathbb{B}} u)(t_0) \right]^{\frac{1}{2}} \\ &- I(0) \geq \frac{1}{2} \beta^2 - I(0) \geq \frac{1}{2} \beta^2 - k(\beta) = \frac{C_*^p C_h}{p} \beta^p. \end{aligned} \quad (3.19)$$

Hence,

$$\|u\|_{\mathcal{H}_{2,0}^{1, \frac{n}{2}}(\mathbb{B})}^p \geq \frac{\beta^p}{l^{\frac{p}{2}}}. \quad (3.20)$$

□

Lemma 3.3. *Suppose that 1.5 holds. Then there exists a positive constant M such that*

$$\|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^s \leq M \left(\|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^p \right), \quad (3.21)$$

for any $u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ and $s \in [2, p]$.

Proof. We consider two cases. First, if $\|u\|_{L^{\frac{n}{p}}(\mathbb{B})} \leq 1$, by Sobolove embedding Theorem and Poincaré's inequality we obtain

$$\begin{aligned} \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^s &\leq \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^2 \leq C_{Poin} \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{p}}(\mathbb{B})}^2 + \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^2 \\ &\leq C_* l^{\frac{1}{2}} C_{Poin} \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^2 \\ &\leq C_* l^{\frac{1}{2}} C_{Poin} \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^p \\ &\leq M \left(\|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^p \right), \end{aligned}$$

where $M > \max\{C_* l^{\frac{1}{2}} C_{Poin}, 1\}$. Now, we assume that $u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ such that $\|u\|_{L^{\frac{n}{p}}(\mathbb{B})} \geq 1$, then

$$\|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^s \leq \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^p \leq M \left(\|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^p \right).$$

□

We set $\mathcal{I}(t) = I_1 - I(t)$ and use positive constant M which depends only on p and l . Here, we prove a result about function $\mathcal{I}(t)$ by using of Lemma 3.3 and 3.2.

Proposition 3.4. *Suppose that u is a weak solution of 1.2 and 1.5 holds. Then, there exists a positive constant $\delta = \delta(l, p)$ such that for any $s \in [2, p]$*

$$\|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^s \leq \Lambda \left[-2\mathcal{I}(t) - \|u_t\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 - (g \circ \nabla_{\mathbb{B}} u)(t) + \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^p \right] \quad (3.22)$$

where, $\Lambda = \Lambda(M, \delta, C_h)$ is a positive constant.

Proof. According to assumption A_1 and relation 3.1 one can get

$$\begin{aligned}
\frac{1}{2}(1-l)\|\nabla_{\mathbb{B}}u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 &\leq \frac{1}{2}(1-\int_0^t g(s)ds)\|\nabla_{\mathbb{B}}u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 \\
&\leq I(t) - \frac{1}{2}\|u_t\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 - \frac{1}{2}(g \circ \nabla_{\mathbb{B}}u)(t) + \frac{1}{p} \int_{\mathbb{B}} h(x)|u|^p \frac{dx_1}{x_1} dx' \\
&\leq I_1 - \mathcal{I}(t) - \frac{1}{2}\|u_t\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 - \frac{1}{2}(g \circ \nabla_{\mathbb{B}}u)(t) \\
&\quad + \frac{1}{p} \int_{\mathbb{B}} h(x)|u|^p \frac{dx_1}{x_1} dx' \\
&\leq I_1 - \mathcal{I}(t) - \frac{1}{2}\|u_t\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 - \frac{1}{2}(g \circ \nabla_{\mathbb{B}}u)(t) + \frac{C_h}{p}\|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^p. \quad (3.23)
\end{aligned}$$

On the other hand, by relations 3.14 and 3.20, we obtain the following estimation for I_1 ,

$$\begin{aligned}
I_1 &= \left(\frac{1}{C_*^p C_h} \right)^{\frac{2}{p-2}} \left[\frac{1}{2} - \frac{C_*^p C_h}{p} \right] \leq \beta^2 \left[\frac{1}{2} - \frac{C_*^p C_h}{p} \right] \\
&\leq \frac{1}{l} \left[\frac{1}{2} - \frac{C_*^p C_h}{p} \right] \|u\|_{\mathcal{H}_2^{1, \frac{n}{2}}(\mathbb{B})}^2 \\
&= \frac{1}{l} \left[\frac{1}{2} - \frac{C_*^p C_h}{p} \right] \|\nabla_{\mathbb{B}}u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2. \quad (3.24)
\end{aligned}$$

From 3.24, we get

$$\begin{aligned}
\frac{1}{2}(1-l)\|\nabla_{\mathbb{B}}u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 &\leq \frac{1}{l} \left[\frac{1}{2} - \frac{C_*^p C_h}{p} \right] \|\nabla_{\mathbb{B}}u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 \\
&\quad - \mathcal{I}(t) - \frac{1}{2}\|u_t\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 - \frac{1}{2}(g \circ \nabla_{\mathbb{B}}u)(t) + \frac{C_h}{p}\|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^p. \quad (3.25)
\end{aligned}$$

Hence, by relation 3.25, one can get

$$\left[\frac{2C_*^p C_h - p(l^2 - l + 1)}{2lp} \right] \|\nabla_{\mathbb{B}}u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 \leq -\mathcal{I}(t) - \frac{1}{2}\|u_t\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 - \frac{1}{2}(g \circ \nabla_{\mathbb{B}}u)(t) + \frac{C_h}{p}\|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^p.$$

Then,

$$\|\nabla_{\mathbb{B}}u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 \leq \frac{1}{\delta(l, p)} \left[-\mathcal{I}(t) - \frac{1}{2}\|u_t\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 - \frac{1}{2}(g \circ \nabla_{\mathbb{B}}u)(t) + \frac{C_h}{p}\|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^p \right], \quad (3.26)$$

where $\delta = \delta(l, p) = \frac{2C_*^p C_h - p(l^2 - l + 1)}{2lp}$.

Now, we apply Lemma 3.3, then

$$\begin{aligned}
\|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^s &\leq \frac{2}{\delta M} \left[-2\mathcal{I}(t) - \|u_t\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 - (g \circ \nabla_{\mathbb{B}}u)(t) \right] \\
&\quad + \left(M + \frac{C_h}{\delta M p} \right) \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^p. \quad (3.27)
\end{aligned}$$

It follows from 3.27 that

$$\|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^s \leq \Lambda \left[-2\mathcal{I}(t) - \|u_t\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 - (g \circ \nabla_{\mathbb{B}}u)(t) + \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^p \right], \quad (3.28)$$

where

$$\Lambda = \Lambda(M, \delta, C_h) = \max\left\{\frac{2}{\delta M}, M + \frac{C_h}{\delta p M}\right\} \quad (3.29)$$

4. PROOF OF MAIN RESULTS

Proof of Theorem 1.1

According to relation 3.1 and definition of $\mathcal{I}(t)$, we get that

$$\begin{aligned} 0 < \mathcal{I}(0) &= I_1 - I(0) \leq I_1 - I(t) = \mathcal{I}(t) \\ &= I_1 - \left[\frac{1}{2} \|u_t\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + \frac{1}{2} \left(1 - \int_0^t g(s) ds\right) \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 \right. \\ &\quad \left. + \frac{1}{2} (g \circ \nabla_{\mathbb{B}} u)(t) - \frac{1}{p} \int_{\mathbb{B}} h(x) |u|^p \frac{dx_1}{x_1} \right] \leq I_1 - \frac{1}{2} \left[\|u_t\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 \right. \\ &\quad \left. + \frac{1}{2} \left(1 - \int_0^t g(s) ds\right) \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + \frac{1}{2} (g \circ \nabla_{\mathbb{B}} u)(t) \right] + \frac{C_h}{p} \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^p. \end{aligned} \quad (4.1)$$

From Lemma 3.2, for any $t \in [0, \infty)$,

$$\begin{aligned} I_1 &- \frac{1}{2} \left[\|u_t\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + \frac{1}{2} \left(1 - \int_0^t g(s) ds\right) \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + \frac{1}{2} (g \circ \nabla_{\mathbb{B}} u)(t) \right] \\ &< \alpha^2 \left(\frac{1}{2} - \frac{C_*^p C_h}{p} \right) - \frac{1}{2} \beta^2 = -\frac{\beta^2 C_*^p C_h}{p} < 0. \end{aligned} \quad (4.2)$$

Therefore,

$$0 < \mathcal{I}(0) \leq \mathcal{I}(t) \leq \frac{C_h}{p} \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^p \quad \forall t \geq 0. \quad (4.3)$$

Now, we consider

$$0 < \epsilon \leq \frac{m(1-k)C_*C_{Poin}^m}{N(m-1)},$$

and define

$$\mathcal{F}(t) := \mathcal{I}^{1-k}(t) + \epsilon \int_{\mathbb{B}} u u_t \frac{dx_1}{x_1} dx', \quad (4.4)$$

such that

$$0 < k \leq \min\left\{\frac{p-2}{2p}, \frac{p-m}{p(m-1)}\right\}.$$

Moreover,

$$\mathcal{F}(0) = \mathcal{I}^{1-k}(0) + \epsilon \int_{\mathbb{B}} u_0(x) u_1(x) \frac{dx_1}{x_1} dx' > 0.$$

We take a derivative from 4.4 and use 1.2, then

$$\begin{aligned}
\mathcal{F}'(t) &= (1-k)\mathcal{I}^{-k}(t)\mathcal{I}'(t) + \epsilon \int_{\mathbb{B}} [u_t^2 + uu_{tt}] \frac{dx_1}{x_1} dx' \\
&= (1-k)\mathcal{I}^{-k}(t) \left[\int_{\mathbb{B}} f(x)|u_t|^m \frac{dx_1}{x_1} dx' - \frac{1}{2} \int_0^t g'(t-\tau) \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u(t) - \nabla_{\mathbb{B}} u(\tau)|^2 \frac{dx_1}{x_1} dx' d\tau \right. \\
&\quad \left. + \frac{1}{2} g(t) \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 \right] \\
&\quad + \epsilon \int_{\mathbb{B}} \left[u_t^2 + h(x)|u|^p - f(x)u_t|u_t|^{m-2}u - \int_0^t g(t-\tau)\Delta_{\mathbb{B}} u(\tau)u d\tau + (\Delta_{\mathbb{B}} u)u \right] \frac{dx_1}{x_1} dx' \\
&= (1-k)\mathcal{I}^{-k}(t) \left[\int_{\mathbb{B}} f(x)|u_t|^m \frac{dx_1}{x_1} dx' - \frac{1}{2} \int_0^t g'(t-\tau) \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u(t) - \nabla_{\mathbb{B}} u(\tau)|^2 \frac{dx_1}{x_1} dx' d\tau \right. \\
&\quad \left. + \frac{1}{2} g(t) \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 \right] + \epsilon \int_{\mathbb{B}} [u_t^2 - |\nabla_{\mathbb{B}} u|^2] \frac{dx_1}{x_1} dx' \\
&\quad + \epsilon \int_0^t g(t-\tau) \int_{\mathbb{B}} \nabla_{\mathbb{B}} u(t) \cdot \nabla_{\mathbb{B}} u(\tau) \frac{dx_1}{x_1} dx d\tau + \epsilon \int_{\mathbb{B}} h(x)|u|^p \frac{dx_1}{x_1} dx' \\
&\quad - \epsilon \int_{\mathbb{B}} f(x)|u_t|^{m-2} u_t u \frac{dx_1}{x_1} dx'. \tag{4.5}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{F}'(t) &\geq (1-k)\mathcal{I}^{-k}(t) \int_{\mathbb{B}} f(x)|u_t|^m \frac{dx_1}{x_1} dx' + \epsilon \int_{\mathbb{B}} [u_t^2 - |\nabla_{\mathbb{B}} u|^2] \frac{dx_1}{x_1} dx' \\
&\quad + \epsilon \int_{\mathbb{B}} h(x)|u|^p \frac{dx_1}{x_1} dx' - \epsilon \int_{\mathbb{B}} f(x)|u_t|^{m-2} u_t u \frac{dx_1}{x_1} dx' + \epsilon \int_0^t g(t-\tau) \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 d\tau \\
&\quad + \epsilon \int_0^t g(t-\tau) \int_{\mathbb{B}} \nabla_{\mathbb{B}} u(t) \cdot [\nabla_{\mathbb{B}} u(\tau) - \nabla_{\mathbb{B}} u(t)] \frac{dx_1}{x_1} dx' d\tau. \tag{4.6}
\end{aligned}$$

Now, we apply the Schwartz inequality, then we obtain

$$\begin{aligned}
\mathcal{F}'(t) &\geq (1-k)\mathcal{I}^{-k}(t) \int_{\mathbb{B}} f(x)|u_t|^m \frac{dx_1}{x_1} dx' + \epsilon \int_{\mathbb{B}} [u_t^2 - |\nabla_{\mathbb{B}} u|^2] \frac{dx_1}{x_1} dx' \\
&\quad + \epsilon \int_{\mathbb{B}} h(x)|u|^p \frac{dx_1}{x_1} dx' - \epsilon \int_{\mathbb{B}} f(x)|u_t|^{m-2} u_t u \frac{dx_1}{x_1} dx' + \epsilon \int_0^t g(t-\tau) \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 d\tau \\
&\quad - \epsilon \int_0^t g(t-\tau) \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})} \|\nabla_{\mathbb{B}} u(\tau) - \nabla_{\mathbb{B}} u(t)\|_{L^{\frac{n}{2}}(\mathbb{B})} d\tau. \tag{4.7}
\end{aligned}$$

On the other hand, we utilize Young's inequality to estimate the last term on the right hand side of 4.7 and use the definition of $I(t)$ to substitute for $\int_{\mathbb{B}} h(x)|u|^p \frac{dx_1}{x_1} dx'$. Hence,

$$\begin{aligned}
\mathcal{F}'(t) &\geq (1-k)\mathcal{I}^{-k}(t) \int_{\mathbb{B}} f(x)|u_t|^m \frac{dx_1}{x_1} dx' + \epsilon \int_{\mathbb{B}} u_t^2 \frac{dx_1}{x_1} dx' - \epsilon \left(1 - \int_0^t g(s)ds\right) \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 \\
&+ \epsilon \left(\frac{p}{2} \|u_t\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + \frac{p}{2} (g \circ \nabla_{\mathbb{B}} u)(t) + \frac{p}{2} [1 - \int_0^t g(s)ds] \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 - pI(t) \right) \\
&- \epsilon \int_{\mathbb{B}} f(x)|u_t|^{m-2} u_t u \frac{dx_1}{x_1} dx' - \epsilon \tau (g \circ \nabla_{\mathbb{B}} u)(t) - \frac{\epsilon}{4\tau} \int_0^t g(s)ds \|\nabla_{\mathbb{B}} u(t)\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 \\
&\geq (1-k)\mathcal{I}^{-k}(t) \int_{\mathbb{B}} f(x)|u_t|^m \frac{dx_1}{x_1} dx' + \epsilon \int_{\mathbb{B}} u_t^2 \frac{dx_1}{x_1} dx' - \epsilon \left(1 - \int_0^t g(s)ds\right) \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 \\
&+ \epsilon \left(\frac{p}{2} \|u_t\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + \frac{p}{2} (g \circ \nabla_{\mathbb{B}} u)(t) + \frac{p}{2} [1 - \int_0^t g(s)ds] \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + p\mathcal{I}(t) - pI_1 \right) \\
&- \epsilon \int_{\mathbb{B}} f(x)|u_t|^{m-2} u_t u \frac{dx_1}{x_1} dx' - \epsilon \tau (g \circ \nabla_{\mathbb{B}} u)(t) - \frac{\epsilon}{4\tau} \int_0^t g(s)ds \|\nabla_{\mathbb{B}} u(t)\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 \\
&\geq (1-k)\mathcal{I}^{-k}(t) \int_{\mathbb{B}} f(x)|u_t|^m \frac{dx_1}{x_1} dx' + \epsilon(1 + \frac{p}{2}) \int_{\mathbb{B}} u_t^2 \frac{dx_1}{x_1} dx' + \epsilon p\mathcal{I}(t) \\
&+ \epsilon(\frac{p}{2} - \tau) (g \circ \nabla_{\mathbb{B}} u)(t) - \epsilon \int_{\mathbb{B}} f(x)|u_t|^{m-2} u_t u \frac{dx_1}{x_1} dx' \\
&+ \epsilon \left[(\frac{p}{2} - 1) - (\frac{p}{2} - 1 + \frac{1}{4\tau}) \int_0^\infty g(s)ds \right] \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2, \tag{4.8}
\end{aligned}$$

for some $0 < \tau < \frac{p}{2}$.

Therefore,

$$\begin{aligned}
\mathcal{F}'(t) &\geq (1-k)\mathcal{I}^{-k}(t) \int_{\mathbb{B}} f(x)|u_t|^m \frac{dx_1}{x_1} dx' + \epsilon(1 + \frac{p}{2}) \int_{\mathbb{B}} u_t^2 \frac{dx_1}{x_1} dx' + \epsilon p\mathcal{I}(t) \\
&+ \epsilon M_1 (g \circ \nabla_{\mathbb{B}} u)(t) + \epsilon M_2 \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 - \epsilon \int_{\mathbb{B}} f(x)|u_t|^{m-2} u_t u \frac{dx_1}{x_1} dx', \tag{4.9}
\end{aligned}$$

where $M_1 := \frac{p}{2} - \tau$ and $M_2 := (\frac{p}{2} - 1) - (\frac{p}{2} - 1 + \frac{1}{4\tau}) \int_0^\infty g(s)ds$ are positive constants.

To estimate of the last term in 4.9, we exploit Young's inequality as follows:

$$\begin{aligned}
\int_{\mathbb{B}} f(x)|u_t|^{m-2} u_t u \frac{dx_1}{x_1} dx' &\leq \int_{\mathbb{B}} |f(x)| |u| |u_t|^{m-1} \frac{dx_1}{x_1} dx' \\
&\leq \left[\frac{\theta^m C_f^m}{m} \int_{\mathbb{B}} |u|^m \frac{dx_1}{x_1} dx' + \frac{m-1}{m} \theta^{-\frac{m}{m-1}} \int_{\mathbb{B}} |u_t|^m \frac{dx_1}{x_1} dx' \right] \\
&\leq \left[\frac{\theta^m C_f^m}{m} \|u\|_{L^{\frac{n}{m}}(\mathbb{B})}^m + \frac{m-1}{m} \theta^{-\frac{m}{m-1}} \|u_t\|_{L^{\frac{n}{m}}(\mathbb{B})}^m \right]. \tag{4.10}
\end{aligned}$$

Now, we insert 4.10 in 4.9, then we get

$$\begin{aligned}
\mathcal{F}'(t) &\geq (1-k)\mathcal{I}^{-k}(t) \int_{\mathbb{B}} f(x)|u_t|^m \frac{dx_1}{x_1} dx' + \epsilon(1 + \frac{p}{2}) \int_{\mathbb{B}} u_t^2(x, t) \frac{dx_1}{x_1} dx' \\
&+ \epsilon p \mathcal{I}(t) + \epsilon M_1(g \circ \nabla_{\mathbb{B}} u)(t) + \epsilon M_2 \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 - \frac{\epsilon \theta^m C_f^m}{m} \|u\|_{L^{\frac{n}{m}}(\mathbb{B})}^m \\
&- \frac{\epsilon(m-1)\theta^{-\frac{m}{m-1}}}{m} \|u_t\|_{L^{\frac{n}{m}}(\mathbb{B})}^m.
\end{aligned} \tag{4.11}$$

Sine our itegral is taken over the variable $x = (x_1, x')$, we can consider the parametre θ . Hence, we consider $\theta^{-\frac{m}{m-1}} = N \mathcal{I}^{-k}(t)$ for large enough N . Now, we apply this equality in relation 4.11 and by relation 1.4 obtain

$$\begin{aligned}
\mathcal{F}'(t) &\geq (1-k)\mathcal{I}^{-k}(t) C^* \|\nabla_{\mathbb{B}} u_t\|_{L^{\frac{n}{m}}(\mathbb{B})}^m + \epsilon(1 + \frac{p}{2}) \int_{\mathbb{B}} u_t^2(x, t) \frac{dx_1}{x_1} dx' \\
&+ \epsilon p \mathcal{I}(t) + \epsilon M_1(g \circ \nabla_{\mathbb{B}} u)(t) + \epsilon M_2 \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 - \frac{\epsilon N^{1-m} C_f^m}{m} \mathcal{I}^{k(m-1)}(t) \|u\|_{L^{\frac{n}{m}}(\mathbb{B})}^m \\
&- \frac{\epsilon(m-1)N}{m} \mathcal{I}^{-k}(t) \|u_t\|_{L^{\frac{n}{m}}(\mathbb{B})}^m = (1-k)\mathcal{I}^{-k}(t) C^* C_{Poin}^m \|u_t\|_{L^{\frac{n}{m}}(\mathbb{B})}^m \\
&+ \epsilon(1 + \frac{p}{2}) \int_{\mathbb{B}} u_t^2(x, t) \frac{dx_1}{x_1} dx' \\
&+ \epsilon p \mathcal{I}(t) + \epsilon M_1(g \circ \nabla_{\mathbb{B}} u)(t) + \epsilon M_2 \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 - \frac{\epsilon N^{1-m} C_f^m}{m} \mathcal{I}^{k(m-1)}(t) \|u\|_{L^{\frac{n}{m}}(\mathbb{B})}^m \\
&- \frac{\epsilon(m-1)N}{m} \mathcal{I}^{-k}(t) \|u_t\|_{L^{\frac{n}{m}}(\mathbb{B})}^m \\
&= \left[(1-k) C^* C_{Poin}^m - \frac{\epsilon N(m-1)}{m} \right] \mathcal{I}^{-k}(t) \|u_t\|_{L^{\frac{n}{m}}(\mathbb{B})}^m + \epsilon(1 + \frac{p}{2}) \int_{\mathbb{B}} u_t^2(x, t) \frac{dx_1}{x_1} dx' \\
&+ \epsilon p \mathcal{I}(t) + \epsilon M_1(g \circ \nabla_{\mathbb{B}} u)(t) + \epsilon M_2 \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 \\
&+ \epsilon \left[p \mathcal{I}(t) - \frac{N^{1-m} C_f^m}{m} \mathcal{I}^{k(m-1)}(t) \|u\|_{L^{\frac{n}{m}}(\mathbb{B})}^m \right].
\end{aligned} \tag{4.12}$$

Now, we utilize relation 4.3 and embedding inequality $\|u\|_{L^{\frac{n}{m}}(\mathbb{B})}^m \leq C_{emb} \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^m$,

$$\mathcal{I}^{k(m-1)}(t) \|u\|_{L^{\frac{n}{m}}(\mathbb{B})}^m \leq \left(\frac{C_h}{p} \right)^{k(m-1)} C_{emb} \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^{m+kp(m-1)}.$$

Therefore, it follows from 4.12 and Lemma 3.3 for $s = m + kp(m-1) \leq p$,

$$\begin{aligned}
\mathcal{F}'(t) &\geq \left[(1-k)C^*C_{Poin}^m - \frac{\epsilon N(m-1)}{m} \right] \mathcal{I}^{-k}(t) \|u_t\|_{L^{\frac{m}{m}}(\mathbb{B})}^m + \epsilon(1 + \frac{p}{2}) \int_{\mathbb{B}} u_t^2(x, t) \frac{dx_1}{x_1} dx' \\
&+ \epsilon p \mathcal{I}(t) + \epsilon M_1(g \circ \nabla_{\mathbb{B}} u)(t) + \epsilon M_2 \|\nabla_{\mathbb{B}} u\|_{L^{\frac{p}{2}}(\mathbb{B})}^2 \\
&+ \epsilon \left[p \mathcal{I}(t) - \frac{N^{1-m} C_f^m}{m} \mathcal{I}^{k(m-1)}(t) \|u\|_{L^{\frac{m}{m}}(\mathbb{B})}^m \right] \\
&= \left[(1-k)C^*C_{Poin}^m - \frac{\epsilon N(m-1)}{m} \right] \mathcal{I}^{-k}(t) \|u_t\|_{L^{\frac{m}{m}}(\mathbb{B})}^m + \epsilon(1 + \frac{p}{2}) \int_{\mathbb{B}} u_t^2(x, t) \frac{dx_1}{x_1} dx' \\
&+ \epsilon \left[p \mathcal{I}(t) - \frac{N^{1-m} C_f^m C_{emb}}{m} \left(\frac{C_h}{p} \right)^{k(m-1)} \left\{ -2\mathcal{I}(t) - \|u_t\|_{L^{\frac{p}{2}}(\mathbb{B})}^2 \right. \right. \\
&- \left. \left. (g \circ \nabla_{\mathbb{B}} u)(t) + \|u\|_{L^{\frac{p}{p}}(\mathbb{B})}^p \right\} \right] \\
&\geq \left[(1-k)C^*C_{Poin}^m - \frac{\epsilon N(m-1)}{m} \right] \mathcal{I}^{-k}(t) \|u_t\|_{L^{\frac{m}{m}}(\mathbb{B})}^m + \epsilon(1 + \frac{p}{2} + R) \|u_t\|_{L^{\frac{p}{2}}(\mathbb{B})}^2 \\
&+ \epsilon(p + 2R)\mathcal{I}(t) + \epsilon(M_1 + R)(g \circ \nabla_{\mathbb{B}} u)(t) \\
&+ \epsilon M_2 \|\nabla_{\mathbb{B}} u\|_{L^{\frac{p}{2}}(\mathbb{B})}^2 - \epsilon R \|u\|_{L^{\frac{p}{p}}(\mathbb{B})}^p, \tag{4.13}
\end{aligned}$$

where $R := \frac{N^{1-m} C_f^m C_{emb}}{m} \left(\frac{C_h}{p} \right)^{k(m-1)}$ is a positive constant. We set $M-3 < \min\{M_1, M_2, \frac{p}{2}\}$.

Moreover, we can get the following estimations for $p = 2M_3 + (p - 2M_3)$:

$$\begin{aligned}
\mathcal{F}'(t) &\geq \left[(1-k)C^*C_{Poin}^m - \frac{\epsilon N(m-1)}{m} \right] \mathcal{I}^{-k}(t) \|u_t\|_{L^{\frac{m}{m}}(\mathbb{B})}^m + \epsilon(1 + \frac{p}{2} + R - M_3) \|u_t\|_{L^{\frac{p}{2}}(\mathbb{B})}^2 \\
&+ \epsilon(p + 2R - 2M_3)\mathcal{I}(t) + \epsilon(M_1 + R - M_3)(g \circ \nabla_{\mathbb{B}} u)(t) \\
&+ \epsilon(M_2 - M_3) \|\nabla_{\mathbb{B}} u\|_{L^{\frac{p}{2}}(\mathbb{B})}^2 - \epsilon \left(\frac{2M_3}{p} - R \right) \|u\|_{L^{\frac{p}{p}}(\mathbb{B})}^p. \tag{4.14}
\end{aligned}$$

For large enough N , we take

$$\gamma := \min \left\{ \left(1 + \frac{p}{2} + R - M_3\right), (p + 2R - 2M_3), (M_1 + R - M_3), (M_2 - M_3) \right\}.$$

Then,

$$\begin{aligned}
\mathcal{F}'(t) &\geq \left[(1-k)C^*C_{Poin}^m - \frac{\epsilon N(m-1)}{m} \right] \mathcal{I}^{-k}(t) \|u_t\|_{L^{\frac{m}{m}}(\mathbb{B})}^m \\
&+ \epsilon \gamma \left[\mathcal{I}(t) + \|u_t\|_{L^{\frac{p}{2}}(\mathbb{B})}^2 + \|u\|_{L^{\frac{p}{p}}(\mathbb{B})}^p + (g \circ \nabla_{\mathbb{B}} u)(t) \right]. \tag{4.15}
\end{aligned}$$

On the other hand, since $\mathcal{I}(t) \geq \mathcal{I}(0) > 0$, it follows that

$$\mathcal{F}(t) \geq \mathcal{F}(0) > 0 \quad \forall t \geq 0.$$

Therefore,

$$\mathcal{F}'(t) \geq \epsilon \gamma \left[\mathcal{I}(t) + \|u_t\|_{L^{\frac{p}{2}}(\mathbb{B})}^2 + \|u\|_{L^{\frac{p}{p}}(\mathbb{B})}^p + (g \circ \nabla_{\mathbb{B}} u)(t) \right]. \tag{4.16}$$

Furthermore, by Hölder's inequality and embedding map one can get

$$\left| \int_{\mathbb{B}} u u_t \frac{dx_1}{x_1} dx' \right| \leq \|u\|_{L^{\frac{n}{2}}(\mathbb{B})} \|u_t\|_{L^{\frac{n}{2}}(\mathbb{B})} \leq C_{emb} \|u\|_{L^{\frac{n}{p}}(\mathbb{B})} \|u_t\|_{L^{\frac{n}{2}}(\mathbb{B})}. \quad (4.17)$$

Hence,

$$\left| \int_{\mathbb{B}} u u_t \frac{dx_1}{x_1} dx' \right|^{\frac{1}{1-k}} \leq C_{emb}^{\frac{1}{1-k}} \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^{\frac{1}{1-k}} \|u_t\|_{L^{\frac{n}{2}}(\mathbb{B})}^{\frac{1}{1-k}}. \quad (4.18)$$

We exploit Young's inequality, thus

$$\left| \int_{\mathbb{B}} u u_t \frac{dx_1}{x_1} dx' \right|^{\frac{1}{1-k}} \leq C_{emb}^{\frac{1}{1-k}} \left(\frac{D^{\frac{\mu}{1-k}}}{\mu} \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^{\frac{\mu}{1-k}} + \frac{D^{-\frac{\nu}{1-k}}}{\nu} \|u_t\|_{L^{\frac{n}{2}}(\mathbb{B})}^{\frac{\nu}{1-k}} \right), \quad (4.19)$$

such that $\frac{1}{\mu} + \frac{1}{\nu} = 1$. In fact, we consider $\frac{\mu}{1-k} = \frac{2}{(1-2k)} \leq p$, then $\nu = 2(1-k)$.

Therefore,

$$\left| \int_{\mathbb{B}} u u_t \frac{dx_1}{x_1} dx' \right|^{\frac{1}{1-k}} \leq A \left\{ \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^s + \|u_t\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 \right\}, \quad (4.20)$$

where $s = \frac{2}{(1-2k)} \leq p$ and $A := \max\{C_{emb}^{\frac{1}{1-k}} D^{\frac{\mu}{1-k}}, C_{emb}^{\frac{1}{1-k}} D^{-\frac{\nu}{1-k}}\}$. Again, we apply Lemma 3.3,

$$\begin{aligned} \left| \int_{\mathbb{B}} u u_t \frac{dx_1}{x_1} dx' \right|^{\frac{1}{1-k}} &\leq A \left\{ (1-\Lambda) \|u_t\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + \Lambda \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^p - \Lambda (g \circ \nabla_{\mathbb{B}} u)(t) - 2\Lambda \mathcal{I}(t) \right\} \\ &\leq G \left\{ \mathcal{I}(t) + \|u_t\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + (g \circ \nabla_{\mathbb{B}} u)(t) + \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^p \right\}, \end{aligned} \quad (4.21)$$

Where $G := \max\{A(1-\Lambda), 2\Lambda\}$ is a positive constant. Then, for every $t \geq 0$ one can obtain

$$\begin{aligned} \mathcal{F}^{\frac{1}{1-k}}(t) &= \left(\mathcal{I}^{1-k}(t) + \epsilon \int_{\mathbb{B}} u_t(x, t) u(x, t) \frac{dx_1}{x_1} dx' \right)^{\frac{1}{1-k}} \\ &\leq 2^{\frac{1}{1-k}} \left(\mathcal{I}(t) + \left| \int_{\mathbb{B}} u u_t \frac{dx_1}{x_1} dx' \right|^{\frac{1}{1-k}} \right) \\ &\leq 2^{\frac{1}{1-k}} \left(\mathcal{I}(t) + G \left\{ \mathcal{I}(t) + \|u_t\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + (g \circ \nabla_{\mathbb{B}} u)(t) + \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^p \right\} \right) \\ &\leq \Gamma \left\{ \mathcal{I}(t) + \|u_t\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + (g \circ \nabla_{\mathbb{B}} u)(t) + \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^p \right\}, \end{aligned} \quad (4.22)$$

where $\Gamma = \max\{2^{\frac{1}{1-k}}, 2^{\frac{1}{1-k}} G\}$ is a positive constant. One can use relations 4.16 and 4.22 and then obtains for any $t \geq 0$ that

$$\mathcal{F}'(t) \geq \Omega \mathcal{F}^{\frac{1}{1-k}}(t), \quad (4.23)$$

where Ω is a positive constan depending only $\epsilon\gamma$ and Γ . Now, we take integral from 4.23 over interval $(0, t)$ and get

$$\mathcal{F}^{\frac{1}{1-k}}(t) \geq \frac{1}{\mathcal{F}^{-\frac{k}{1-k}}(0) - \Omega t^{\frac{k}{1-k}}}. \quad (4.24)$$

Hence, it follows from 4.24 that $\mathcal{F}(t)$ blows up in time

$$T(k, \Omega) = T \leq \frac{1-k}{\Omega k (\mathcal{F}(0))^{\frac{k}{1-k}}}. \quad (4.25)$$

□

Here, we investigate the lower bound of the blow up time for the blow up solution of problem 1.2.

Proof of Theorem 1.2

Proof. First, we define $\mathcal{H}(t) = \int_{\mathbb{B}} h(x) |u(x, t)|^p \frac{dx_1}{x_1} dx'$. Then,

$$\begin{aligned} \mathcal{H}'(t) &= p \int_{\mathbb{B}} |u(x, t)|^{p-2} u(x, t) u_t(x, t) \frac{dx_1}{x_1} dx' \\ &\leq \frac{p C_h}{2} \left(\int_{\mathbb{B}} |u(x, t)|^{2(p-1)} \frac{dx_1}{x_1} dx' + \int_{\mathbb{B}} |u_t(x, t)|^2 \frac{dx_1}{x_1} dx' \right) \end{aligned} \quad (4.26)$$

To estimate the first term on the right side of inequality 4.26, we consider the following two cases. In the first case, we consider $2 < p \leq 2^*$. Suppose that $q = 2(p-1)$ and $r = n(p-2)$. Using of Hölder's inequality and embedding Theorem, one can get

$$\int_{\mathbb{B}} |u(x, t)|^q \frac{dx_1}{x_1} dx' = \int_{\mathbb{B}} |u|^{q\eta} |u|^{q(1-\eta)} \frac{dx_1}{x_1} dx' \leq \left(\int_{\mathbb{B}} |u|^r \frac{dx_1}{x_1} dx' \right)^{\frac{q\eta}{r}} \left(\int_{\mathbb{B}} |u|^{2^*} \frac{dx_1}{x_1} dx' \right)^{\frac{q(1-\eta)}{2^*}},$$

where η satisfies $\frac{q\eta}{r} + \frac{q(1-\eta)}{2^*} = 1$. A simple calculation shows that

$$\eta = \frac{r(2^* - q)}{q(2^* - r)}, \quad \frac{q\eta}{r} = \frac{2}{n}, \quad \frac{q - q\eta}{2^*} = 1 - \frac{2}{n}.$$

Then, we use the Hölder inequality

$$\|u\|_{L^{\frac{n}{n}}(\mathbb{B})}^{\frac{2r}{n}} \leq |\mathbb{B}|^{\frac{2(p-r)}{np}} \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^{\frac{2r}{n}} \leq \left(1 + |\mathbb{B}|^{\frac{2(p-r)}{np}} \right) \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^{\frac{2r}{n}}$$

and embedding inequality $\|u\|_{L^{\frac{n}{2^*}}(\mathbb{B})} \leq C_* \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}$, where C_* is the best constant of the Sobolev embedding $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \hookrightarrow L^{\frac{n}{2^*}}(\mathbb{B})$.

$$\begin{aligned} \|u\|_{L^{\frac{n}{q}}(\mathbb{B})}^q &\leq \|u\|_{L^{\frac{n}{r}}(\mathbb{B})}^{q\eta} \|u\|_{L^{\frac{n}{2^*}}(\mathbb{B})}^{q(1-\eta)} = \|u\|_{L^{\frac{n}{r}}(\mathbb{B})}^{\frac{2r}{n}} \|u\|_{L^{\frac{n}{2^*}}(\mathbb{B})}^2 \\ &\leq C_*^2 \left(1 + |\mathbb{B}|^{\frac{2(p-r)}{np}} \right) \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^{\frac{2r}{n}} \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 \\ &\leq C_*^2 \left(1 + |\mathbb{B}|^{\frac{2(p-r)}{np}} \right) \left[\|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^{\frac{2r}{n}} + \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^{2t} \right] \\ &\leq C_1 \left(\|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^p + \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 \right)^{k_1}, \end{aligned} \quad (4.27)$$

where $\frac{1}{s} + \frac{1}{t} = 1$, $t := \frac{2r}{np} \cdot s$ and we can deduce $k_1 = \frac{3p-4}{p}$ and $\mathcal{C}_1 = \mathcal{C}_*^2 \left(1 + |\mathbb{B}|^{\frac{2(p-r)}{np}}\right)$.

For the second case, we assume that $\frac{2n}{n-2} < p \leq \frac{2(n-1)}{n-2}$. Then,

$$\begin{aligned} \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^q &\leq \mathcal{C}_*^r \left(1 + |\mathbb{B}|^{1-\frac{q}{2^*}}\right) \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^q \leq \mathcal{C}_*^r \left(1 + |\mathbb{B}|^{1-\frac{q}{2^*}}\right) \left[\|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^q + \|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^{p(p-1)} \right] \\ &\leq \mathcal{C}_2 \left(\|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^p + \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 \right)^{k_2}, \end{aligned} \quad (4.28)$$

where $k_2 = p - 1$, and $\mathcal{C}_2 = \mathcal{C}_*^r \left(1 + |\mathbb{B}|^{1-\frac{q}{2^*}}\right)$ is a positive constant.

From Lemma 3.1

$$\begin{aligned} I(t) \leq I(0) &= \frac{1}{2} \|u_1\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + \|\nabla_{\mathbb{B}} u_0\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 \\ &\quad - \frac{1}{p} \int_{\mathbb{B}} h(x) |u|^p \frac{dx_1}{x_1} \quad \forall t \in [0, T(k, \Omega) = T], \end{aligned} \quad (4.29)$$

where k and Ω are given in proof of Theorem 1.1. Now, using of the definitopn of the functional $I(t)$, assumption A_3 and inequality 4.28 we obtain

$$\begin{aligned} \|u_t(x, t)\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 &+ \frac{1}{(p-1)^2} \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + (g \circ u)(t) \\ &\leq \|u_t(x, t)\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + \left(1 - \int_0^t g(s) ds\right) \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 + (g \circ u)(t) \\ &= \frac{2}{p} \|u_t\|_{L^{\frac{n}{p}}(\mathbb{B})}^p + 2I(t) \leq \frac{2}{p} \mathcal{H}(t) + 2I(0). \end{aligned} \quad (4.30)$$

From computaions 4.26-4.30, we obtain the following inequality:

$$\begin{aligned} \mathcal{H}'(t) &\leq \frac{C_{hp}}{2} \left(\mathcal{C}_i \left[\|u\|_{L^{\frac{n}{p}}(\mathbb{B})}^p + \|\nabla_{\mathbb{B}} u\|_{L^{\frac{n}{2}}(\mathbb{B})}^2 \right] + \frac{2}{p} \mathcal{H}(t) + 2I(0) \right) \\ &\leq \frac{C_{hp}}{2} \left(\mathcal{C}_i \left[\mathcal{H}(t) + \mathcal{C}_0 \left(\frac{2}{p} \mathcal{H}(t) + 2I(0) \right) \right]^{k_i} + \frac{2}{p} \mathcal{H}(t) + 2I(0) \right) \\ &\leq \frac{C_{hp}}{2} \left(\mathcal{C}_i \left[\left(1 + \frac{2\mathcal{C}_0}{p}\right) \mathcal{H}(t) + 2\mathcal{C}_0 I(0) \right]^{k_i} + \frac{2}{p} \mathcal{H}(t) + 2I(0) \right) \\ &\leq \frac{C_{hp}}{2} 2^{k_i-1} \left(\left[1 + \frac{2\mathcal{C}_0}{p}\right]^{k_i} \mathcal{H}^{k_i}(t) + \left(2\mathcal{C}_0 I(0)\right)^{k_i} \right) + \mathcal{H}(t) + pI(0) \\ &= \mathcal{C}_3 \mathcal{H}^{k_i}(t) + \mathcal{H}(t) + \mathcal{C}_4, \end{aligned} \quad (4.31)$$

where $\mathcal{C}_0 = \frac{1}{l}$, $\mathcal{C}_3 = \frac{C_h p \mathcal{C}_i 2^{k_i}}{4} \left(1 + \frac{2\mathcal{C}_0}{p}\right)^{k_i}$ and $\mathcal{C}_4 = pI(0) + \frac{C_h p \mathcal{C}_i}{4} \left(4\mathcal{C}_0 I(0)\right)^{k_i}$ for $i = 1, 2$ are positive constants. We exploit Theorem 1.1 and then get

$$\lim_{t \rightarrow T} \int_{\mathbb{B}} h(x) |u(x, t)|^p \frac{dx_1}{x_1} = +\infty. \quad (4.32)$$

It follows from 4.31 and 4.32,

$$\int_{\mathcal{H}(0)}^{\infty} \frac{1}{\mathcal{C}_3 S^k + S + \mathcal{C}_4} ds \leq T. \quad (4.33)$$

□

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